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A UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR THE SINGULARLY PERTURBED PERIODIC PROBLEM

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ABSTRACT

In this article, a novel numerical scheme is suggested to solve periodical boundary value problem for linear first order singularly perturbed equation. This scheme is constructed by the finite difference method on a special non-uniform mesh (Shishkin mesh) using quadrature rules with the remaining terms in integral form. It is proven that the scheme achieves almost first-order convergence on the discrete maximum norm. Finally, two test problems are considered to demonstrate the accuracy and performance of the method.

Keywords: Singular perturbation, Periodic problem, Finite difference method, Uniform convergence.

1 INTRODUCTION

If a differential equation involves a small positive parameter multiplied by its highest-order derivative, it is called a singularly perturbed differential equation (SPDE). These equations occur as models in science and engineering various fields. For example, they appear in fluid dynamics, chemical reactor theory, enzyme kinetics, quantum mechanics, elasticity, electrical networks, etc. [9]-[11], [15]. In particular, singularly perturbed periodical boundary value problem arises as models in geophysical fluid dynamics [6], in oceanic and atmospheric circulation [11]. Motivated by the above works, we analyze the following problem:

$$\varepsilon u'(t) + a(t)u(t) = f(t), 0 < t < T, \tag{1}$$

$$u(0) = u(T), \tag{2}$$

where $0 < \varepsilon \ll 1$ is the small positive parameter (so called perturbation parameter). Also, $a(t) \geq \alpha > 0, f(t)$ are given functions on $\bar{I} = [0, T]$. We assume that these functions sufficiently smooth (i.e. desired continuous derivatives on $\bar{I}$) and moreover $a(t+T) = a(t), f(t+T) = f(t)$. Furthermore, if $a, f \in C(\bar{I})$, then the problem (1)-(2) has a unique solution $u(t) \in C^1(\bar{I})$, which also satisfies condition (2) (for details see [1]). Studies on the existence and uniqueness of such problems can be found in [1] and references therein.

Since the solutions of SPDEs have initial (or boundary) layers that depend on the ε-parameter, classical numerical methods used to solve such problems are unstable and do not yield accurate results. Therefore, it is necessary to develop appropriate numerical methods that are uniformly convergent with respect to ε. One can also refer [9], [10], [12], [15], for more information about the perturbation theory and layer adapted meshes. Furthermore, in decade years, researchers have studied and improved many aspects of numerical methods for the singularly perturbed problems [3]-[5], [8], [16], [18].

We encounter a few periodic SPDEs studies in the literature, to mention them: Amiraliyev and Duru [2] solved the second-order periodic SPDE by using finite difference scheme on uniform mesh and prove that approximations are first order convergence in the maximum norm. In [7], Cen considered the same problem and proposed a hybrid finite difference scheme on a Shishkin mesh which is proved to be second-order accurate. In [13] the authors proposed cubic spline scheme on variable mesh for solving the same problem. In [14] the authors solved the system of the second-order periodic problem and presented a hybrid scheme using piecewise uniform mesh. These articles involve second-order ones; we did not come across any studies on first-order SPDEs. The motivation for our study is formed by these gaps in the literature.

The main purpose of this paper is to present a uniformly convergent method for solving (1)-(2). This method consists of a fitted difference scheme given on the piecewise uniform mesh (called Shishkin mesh). The scheme is constituted using the finite difference method and appropriate quadrature rules with the terms remaining in the integral form, which is based on the Taylor polynomial whose remaining term is in integral form. Hereby, the truncation errors involve only the first-order derivative of the exact solution, making convergence easier to examine. Since the truncation errors of the above methods include two or higher-order derivatives of the exact solution, the presented method is more advantageous.

The problem (1)-(2) has a boundary layer in a small neighborhood of $t = 0$ when $0 < \varepsilon \ll 1$. The proposed method to solve the problem causes instability on the uniform mesh. We used Shishkin mesh, also called piecewise uniform mesh, to improve this instability. Readers can refer to [9,10,15] for more detailed information on this topic.

The arrangement of the paper: In next section we state the analytical results of solution for (1)-(2) and its derivative and we construct a finite difference scheme and introduce the layer-adapted non-uniform meshes. In Section 3, we perform error analysis and give convergence results on these meshes. In Section 4, we present numerical experiments, which confirm the theoretical results. In the final section, we give a summary of the main conclusions.

From now on, C is defined as a general positive constant that is not only independent of ε but also of the mesh parameter N. Moreover, for any $w(t) \in C(\bar{I})$, the norm defined below will be used:

$$\|w\|_\infty = \max_{t\in\bar{I}} |w(t)|.$$

2 MATERIAL AND METHOD

2.1 The Behavior of the Exact Solution

Here, some estimations of the exact solution of (1)-(2) are given that are needed in later sections.

Lemma 2.1. If $a, f \in C^1(\bar{I})$ and $1 - e^{-\frac{\alpha T}{\varepsilon}} \geq \delta > 0$, then the following inequalities

$$\|u\|_\infty \leq (1 + \delta^{-1})\alpha^{-1}\|f\|_\infty, \tag{3}$$

$$|u'(t)| \leq C\left(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right), \tag{4}$$

hold.

Proof. From Eq.(1), we can write,

$$u(t) = u(0)e^{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^t f(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^t a(\tau)d\tau} d\xi. \tag{5}$$

Using condition (2), we obtain

$$u(0) = \frac{\frac{1}{\varepsilon}\int_0^T f(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^T a(\tau)d\tau} d\xi}{1 - e^{-\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau}}. \tag{6}$$

From (5), taking into account $u(0)$ in (6), we have

$$u(t) = \frac{e^{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau}}{1-e^{-\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau}}\frac{1}{\varepsilon}\int_0^T f(\xi)\,e^{-\frac{1}{\varepsilon}\int_\xi^T a(\tau)d\tau}d\xi + \frac{1}{\varepsilon}\int_0^t f(\xi)\,e^{-\frac{1}{\varepsilon}\int_\xi^t a(\tau)d\tau}d\xi.$$

Also, since $a(t) \geq \alpha > 0$, we get

$$1-e^{-\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau} \geq 1-e^{-\frac{\alpha T}{\varepsilon}} \geq \delta > 0.$$

From this inequality, we have

$$|u(t)| \leq \frac{e^{-\frac{\alpha t}{\varepsilon}}}{\delta}\frac{1}{\varepsilon}\int_0^T |f(\xi)|\,e^{-\frac{\alpha}{\varepsilon}(T-\xi)}d\xi + \frac{1}{\varepsilon}\int_0^t |f(\xi)|\,e^{-\frac{\alpha}{\varepsilon}(t-\xi)}d\xi$$

$$\leq \frac{e^{-\frac{\alpha t}{\varepsilon}}}{\delta}\frac{1}{\varepsilon}\|f\|_\infty\int_0^T e^{-\frac{\alpha}{\varepsilon}(T-\xi)}d\xi + \frac{1}{\varepsilon}\|f\|_\infty\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-\xi)}d\xi$$

$$\leq \frac{e^{-\frac{\alpha t}{\varepsilon}}}{\delta}\|f\|_\infty\alpha^{-1}\left(1-e^{-\frac{\alpha T}{\varepsilon}}\right) + \|f\|_\infty\alpha^{-1}\left(1-e^{-\frac{\alpha t}{\varepsilon}}\right),$$

which leads (3).

Now, for (4), if we differentiate Eq. (1) once, we can write

$$\varepsilon u''(t) + a(t)u'(t) = F(t),$$

with

$$F(t) = f'(t) - a'(t)u(t),$$

and from the solution of this equation, for $u'(t)$, we can write

$$u'(t) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^t a(\zeta)d\zeta} + \frac{1}{\varepsilon}\int_0^t F(\xi)\,e^{-\frac{1}{\varepsilon}\int_\xi^t a(\zeta)d\zeta}d\xi.$$

And from here, we have

$$|u'(t)| \leq |u'(0)|e^{-\frac{\alpha t}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^t |F(\xi)|\,e^{-\frac{\alpha}{\varepsilon}(t-\xi)}d\xi. \tag{7}$$

From conditions of Lemma 2.1 and (3), we get $\|F\|_\infty \leq C$. Therefore, from (7), we can write

$$|u'(t)| \leq |u'(0)|e^{-\frac{\alpha t}{\varepsilon}} + \|F\|_\infty\alpha^{-1}\left(1-e^{-\frac{\alpha t}{\varepsilon}}\right).$$

From (1), we can write

$$|u'(0)| \leq \frac{1}{\varepsilon}[|f(0)| + |a(0)||u(0)|].$$

When Eq. (6) is taken into account in this inequality, we obtain

$$|u'(0)| \leq \frac{\delta^{-1}}{\varepsilon}\int_0^T |f(\xi)|\, e^{-\frac{1}{\varepsilon}\int_\xi^T a(\tau)d\tau} d\xi \leq \frac{C}{\varepsilon}.$$

If we consider this last inequality in (7), we can write

$$|u'(t)| \leq |u'(0)|e^{-\frac{\alpha t}{\varepsilon}} + \|F\|_\infty \alpha^{-1}\left(1 - e^{-\frac{\alpha t}{\varepsilon}}\right)$$

$$\leq \frac{C}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}} + C\alpha^{-1}\left(1 - e^{-\frac{\alpha t}{\varepsilon}}\right),$$

which completes the proof.

2.2 The Difference Scheme and the Layer-Adapted Mesh

Let a non-uniform mesh is defined on I as: $\omega = \{0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T\}$ and $\bar{\omega} = \omega \cup \{t_0 = 0\}$. Also, we establish the stepsize $h_i = t_i - t_{i-1}$ for $1 \leq i \leq N$. For simplicity, we will use $v_i = v(t_i)$. We will denote the approximation of $u(t)$ at t_i as y_i. Besides, for $\{v_i\}$, we will use the difference derivative and norm defined on $\bar{\omega}$ as:

$$v_{\bar{t},i} = \frac{v_i - v_{i-1}}{h_i}, \|v\|_{\infty,\bar{\omega}} = \max_{0\leq i\leq N}|v_i|.$$

To obtain numerical approximation for the problem (1)-(2), integrating Eq. (1) over (t_{i-1}, t_i), we obtain

$$\varepsilon u_{\bar{t},i} + a_i u_i = f_i - R_i, 1 \leq i \leq N-1, \tag{8}$$

with remainder term

$$R_i = h_i^{-1}\int_{t_{i-1}}^{t_i} (t - t_{i-1})\frac{d}{dt}[f(t) - a(t)u(t)]dt. \tag{9}$$

If we neglect R_i in (9), we propose the following difference scheme for approximating (1)-(2):

$$\varepsilon y_{\bar{t},i} + a_i y_i = f_i, 1 \leq i \leq N-1, \tag{10}$$

$$y_0 = y_N. \tag{11}$$

For the difference schemes (10)-(11) to be ε-uniformly convergent, we will use the following special mesh (known as Shishkin mesh, shortly S-mesh, for more details see [10]).

The piecewise uniform mesh divides equidistantly each of the intervals $[0,\sigma]$ and $[\sigma,T]$ into $N/2$ subintervals (N is an even positive integer), the transition point σ is denoted as:

$$\sigma = \min\left\{\frac{T}{2}, \alpha^{-1}\varepsilon lnN\right\}, \tag{12}$$

$$\bar{\omega} = \begin{cases} t_i = ih^{(1)}, & 0 \le i \le \frac{N}{2}, \quad h^{(1)} = \frac{2\sigma}{N}, \\ t_i = \sigma + \left(i - \frac{N}{2}\right)h^{(2)}, & \frac{N}{2}+1 \le i \le N, \quad h^{(2)} = \frac{2(T-\sigma)}{N}. \end{cases}$$

3 CONVERGENCE ANALYSIS

In this section, we will investigate the uniform convergence of the presented method. For this, we will define the error function z as follows:

$z_i = y_i - u_i, 1 \le i \le N-1.$

From (8) and (10), we have

$$\varepsilon z_{\bar{t},i} + a_i z_i = -R_i, \tag{13}$$

$$z_0 = z_N, \tag{14}$$

where R_i is described by (9).

Lemma 3.1. Suppose y_i is the solution of (10)-(11) and

$$1 - \frac{1}{[1+\alpha h^{(1)}/\varepsilon]^{N/2}} - \frac{1}{[1+\alpha h^{(2)}/\varepsilon]^{N/2}} \ge \delta_* > 0.$$

Then the estimate

$$\|y\|_{\infty,\bar{\omega}} \le C\|f\|_{\infty,\bar{\omega}} \tag{15}$$

holds.

Proof. From Eq. (10), we can rewrite

$$y_i = q_i y_{i-1} + g_i, \tag{16}$$

where

$$q_i = \frac{\varepsilon}{\varepsilon + a_i h_i}, g_i = \frac{h_i f_i}{\varepsilon + a_i h_i}.$$

When (11) and (16) are considered together, a first-order difference equation is obtained. The solution of this equation is

$$y_i = y_0 Q_i + \sum_{k=1}^{i} g_k Q_{i-k}, \tag{17}$$

where

$$Q_{i-k} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} q_j, & 0 \le k \le i-1. \end{cases}$$

If we write $i = N$ in (17), we get

$$y_N = y_0 Q_N + \sum_{k=1}^{N} g_k Q_{N-k}.$$

If we take into account condition (11) in this equation, we easily obtain

$$y_0 = \frac{\sum_{k=1}^{N} g_k Q_{N-k}}{1 - Q_N}. \tag{18}$$

If we take this value into account in (17), we have

$$y_i = \frac{Q_i}{1 - Q_N} \sum_{k=1}^{N} g_k Q_{N-k} + \sum_{k=1}^{i} g_k Q_{i-k}.$$

From here,

$$1 - Q_N = 1 - \prod_{i=1}^{N} \frac{1}{1 + a_i h_i/\varepsilon} \ge 1 - \prod_{i=1}^{N} \frac{1}{1 + \alpha h_i/\varepsilon},$$

$$\prod_{i=1}^{N} \frac{1}{1 + \alpha h_i/\varepsilon} = \prod_{i=1}^{N/2} \frac{1}{1 + \alpha h^{(1)}/\varepsilon} + \prod_{i=\frac{N}{2}+1}^{N} \frac{1}{1 + \alpha h^{(2)}/\varepsilon}$$

$$= \frac{1}{[1 + \alpha h^{(1)}/\varepsilon]^{N/2}} + \frac{1}{[1 + \alpha h^{(2)}/\varepsilon]^{N/2}}$$

$$1 - Q_N \ge 1 - \frac{1}{[1 + \alpha h^{(1)}/\varepsilon]^{N/2}} - \frac{1}{[1 + \alpha h^{(2)}/\varepsilon]^{N/2}}.$$

It is easily to see that the absolute value of the numerator in (18) is bounded by $C\|f\|_{\infty,\bar{\omega}}$ and the denominator is bounded from below by δ_*, thereby will be

$|y_0| \le C\|f\|_{\infty,\bar{\omega}}$.

Then, if the principle of discrete maximum is applied to the difference operator [17] $\ell_h y_i \equiv \varepsilon y_{\bar{t},i} + a_i y_i = f_i, 1 \le i \le N-1$, to (10) yields

$$|y_i| \le |y_0| + \alpha^{-1}\|f\|_{\infty,\bar{\omega}},$$

which is the complete of proof.

Lemma 3.2. Let z_i be the solution of (13)-(14). Then

$$\|z\|_{\infty,\bar{\omega}} \le C\|R\|_{\infty,\bar{\omega}}.$$

Proof. It evidently follows from (15) by taking $y \equiv z$ and $f \equiv R$.

Lemma 3.3. Under the above assumptions of Lemma 2.1, for the truncation error R, the following estimate holds

$$\|R\|_{\infty,\bar{\omega}} \le CN^{-1}lnN.$$

Proof. From (9), taking into account (3)-(4), we obtain that on an arbitrary mesh:

$$|R_i| \le C\left(h_i + \frac{1}{\varepsilon}\int_{t_{i-1}}^{t_i} e^{-\frac{\alpha t}{\varepsilon}}dt\right)$$

$$\le C\left\{h_i + \alpha^{-1}(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}})\right\}. \qquad (19)$$

And now, we evaluate to R_i in two different cases for σ values in (12). Firstly, let $\sigma = T/2$. Then $\frac{T}{2} < \alpha^{-1}\varepsilon lnN$ and $h^{(1)} = h^{(2)} = T/2$. Take these conditions into account in (19):

$$|R_i| \le C\left\{h_i + \alpha^{-1}(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}})\right\}$$

$$\le C\left\{\frac{T}{N}+\alpha^{-1}\right\} \le C\left\{\frac{2\alpha^{-1}\varepsilon lnN}{N} + \alpha^{-1}\right\} \le CN^{-1}lnN.$$

Secondly, let $\sigma = \alpha^{-1}\varepsilon lnN$. We give estimations for subintervals $[0,\sigma]$ and $[\sigma,T]$. If $t \in [0,\sigma]$, $h^{(1)} = \frac{2\sigma}{N}$. So, inequality (19) can be written as follows:

$$|R_i| \le C\left\{h_i + \alpha^{-1}(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}})\right\}$$

$$\le C\left\{\frac{2\sigma}{N}+\alpha^{-1}\right\} \le C\left\{\frac{2\alpha^{-1}\varepsilon lnN}{N} + \alpha^{-1}\right\} \le CN^{-1}lnN.$$

If $t \in [\sigma,T]$,, $h^{(2)} = \frac{2(T-\sigma)}{N}$ and $t_i = \sigma + (i - N/2)h^{(2)}$. Since

$$e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}} = e^{-\frac{\alpha[\sigma+(i-1-N/2)h^{(2)}]}{\varepsilon}} - e^{-\frac{\alpha[\sigma+(i-N/2)h^{(2)}]}{\varepsilon}}$$

$$= e^{-\frac{\alpha[\alpha^{-1}\varepsilon lnN+(i-1-N/2)h^{(2)}]}{\varepsilon}} - e^{-\frac{\alpha[\alpha^{-1}\varepsilon lnN+(i-N/2)h^{(2)}]}{\varepsilon}}$$

$$= e^{-[lnN+\alpha\varepsilon^{-1}(i-1-N/2)h^{(2)}]} - e^{-[lnN+\alpha\varepsilon^{-1}(i-N/2)h^{(2)}]}$$

$$= e^{-lnN}\left\{e^{-\alpha\varepsilon^{-1}(i-1-N/2)h^{(2)}} - e^{-\alpha\varepsilon^{-1}(i-N/2)h^{(2)}}\right\} \leq CN^{-1},$$

then it follows from (19) that

$$|R_i| \leq C\left\{h_i + \alpha^{-1}(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}})\right\}$$

$$= C\left\{h^{(2)} + \alpha^{-1}(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}})\right\} \leq C\left\{h^{(2)} + \alpha^{-1}N^{-1}\right\} \leq CN^{-1}.$$

Thus, the proof of Lemma 3.3 is complete.

Now, if we take into account the above lemmas, we give the following main convergence results.

Theorem 3.1. Let u be the solution of (1)-(2) and y the solution of (10)-(11). Then under hypotheses of Lemmas 1.1 and 4.1

$$\|y - u\|_{\infty,\bar{\omega}} \leq CN^{-1}lnN.$$

Proof. This follows immediately by combining the previous lemmas. That is, if Lemma 3.3. is taken into account in Lemma 3.2, the proof is complete.

4 ALGORITHM AND NUMERICAL EXPERIMENTS

In this section, we consider an example to demonstrate the robustness and effectiveness of our scheme defined by equations (10)-(11). At first, we solve the periodic problem (10)-(11) in the Algorithm. Later on, two test problems are presented in the Numerical experiments.

4.1 Algorithm

From (14) and (15), we can easily obtain the solution of the difference problem (10)-(11):

$$y_i = y_0 Q_i + \sum_{k=1}^{i} g_k Q_{i-k}, 1 \leq i \leq N-1,$$

$$y_0 = \frac{\sum_{k=1}^{N} g_k Q_{N-k}}{1-Q_N}, y_N = y_0.$$

4.2 Numerical Experiments

Using the above algorithm, we present the following two problems. The present numerical scheme defined by (10)-(11) is used to solve the problems for several values ε and N. The results obtained here are given in the following tables and figures.

Example 4.1. We examine the following test problem:

$$\varepsilon u'(t) + [3 + \sin(2t+1)]u(t) = \cos(2t-1), 0 < t < \pi,$$

$$u(0) = u(\pi).$$

Example 4.2. We examine the following another test problem:

$$\varepsilon u'(t) + [5 - \sin(\pi t+1)]u(t) = \frac{1}{2}\cos(\pi t-1), 0 < t < 2,$$

$$u(0) = u(2).$$

The exact solutions of the test problems are unknown. Therefore, for error estimates and analysis of convergence, we use double mesh principle ([5], [9]). That is, $e^{\varepsilon,N}$ is defined as the two mesh differences for $u(t)$ and e^N the maximum two mesh differences as:

$$e^{\varepsilon,N} = \max_{0\le i\le N}\left|y_i^N - y_{2i}^{2N}\right|, e^N = \max_{\varepsilon} e^{\varepsilon,N}.$$

Also, the convergence rates are defined as:

$$p^N = \log_2(e^N/e^{2N}).$$

The test problem is solved for the $\varepsilon = 2^{-4i}, i = 1,2,\ldots,6$. The two mesh differences and the convergence rates are listed in Tables 1-2.

Table 1. Computed maximum two mesh differences and convergence rates for Example 4.1.

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	0.0008456	0.0005312	0.0003184	0.0001842	0.0001042
	0.671	0.739	0.789	0.822	
2^{-8}	0.0006353	0.0003942	0.0002337	0.0001345	0.0000757
	0.689	0.754	0.797	0.829	
2^{-12}	0.0006226	0.0003856	0.0002284	0.0001314	0.0000739
	0.691	0.755	0.798	0.830	
2^{-16}	0.0006218	0.0003851	0.0002281	0.0001312	0.0000738
	0.691	0.756	0.798	0.830	
2^{-20}	0.0006218	0.0003851	0.0002281	0.0001312	0.0000738
	0.691	0.756	0.798	0.830	
2^{-24}	0.0006218	0.0003850	0.0002281	0.0001312	0.0000738
	0.691	0.756	0.798	0.830	
e^N	0.0008456	0.0005312	0.0003184	0.0001842	0.0001042
p^N	0.671	0.739	0.789	0.822	

Table 2. Computed maximum two mesh differences and convergence rates for Example 4.2.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-4}	0.0011790	0.0007121	0.0004159	0.0002370	0.0001327
	0.727	0.776	0.811	0.837	
2^{-8}	0.0012686	0.0007649	0.0004460	0.0002539	0.0001421
	0.730	0.778	0.813	0.837	
2^{-12}	0.0012743	0.0007683	0.0004479	0.0002550	0.0001427
	0.730	0.778	0.813	0.838	
2^{-16}	0.0012747	0.0007685	0.0004480	0.0002551	0.0001427
	0.730	0.779	0.812	0.838	
2^{-20}	0.0012747	0.0007685	0.0004480	0.0002551	0.0001427
	0.730	0.779	0.812	0.838	
2^{-24}	0.0012747	0.0007685	0.0004480	0.0002551	0.0001427
	0.730	0.779	0.812	0.838	
e^N	0.0012747	0.0007685	0.0004480	0.0002551	0.0001427
p^N	0.730	0.779	0.812	0.838	

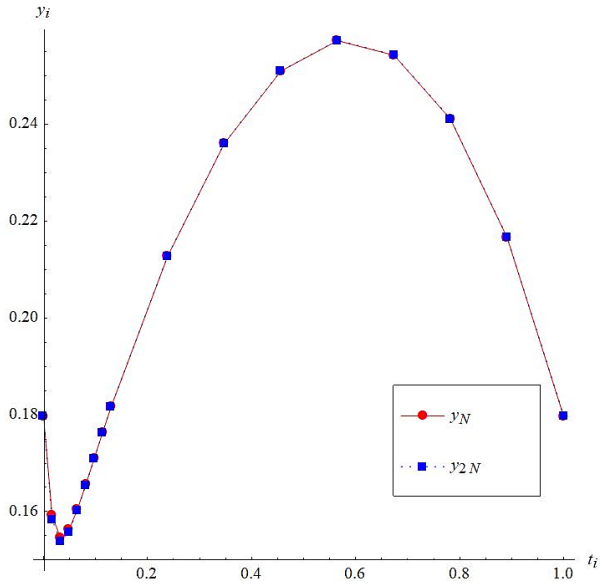


Figure 1. Computational results obtained using two-mesh differences for Example 4.1 ($N = 64$ and $\varepsilon = 2^{-4}$).

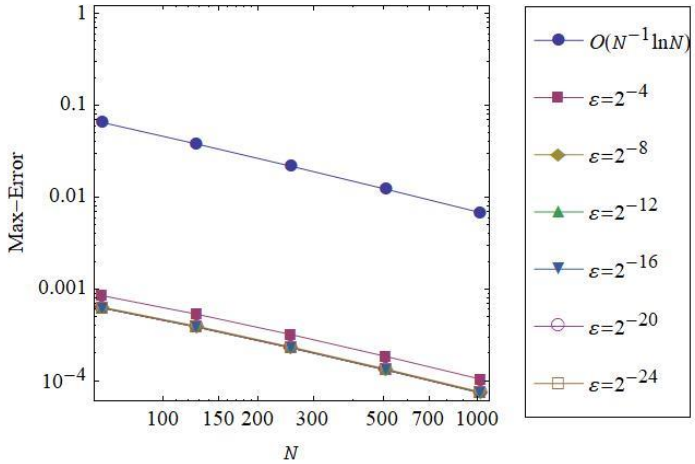


Figure 2. Maximum point-wise errors obtained using various ε and N for Example 4.1.

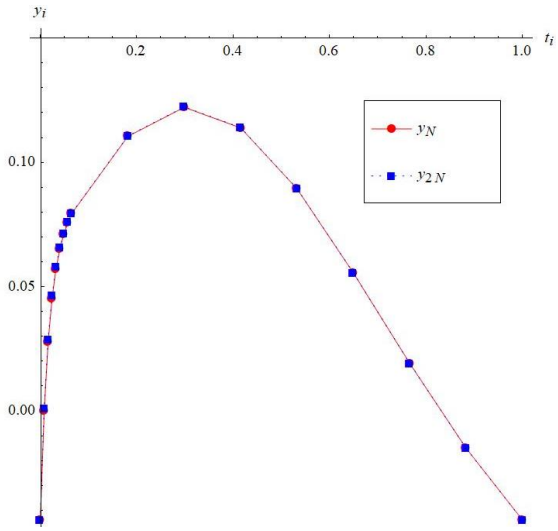


Figure 3. Computational results obtained using two-mesh differences for Example 4.2 ($N = 64$ and $\varepsilon = 2^{-4}$).

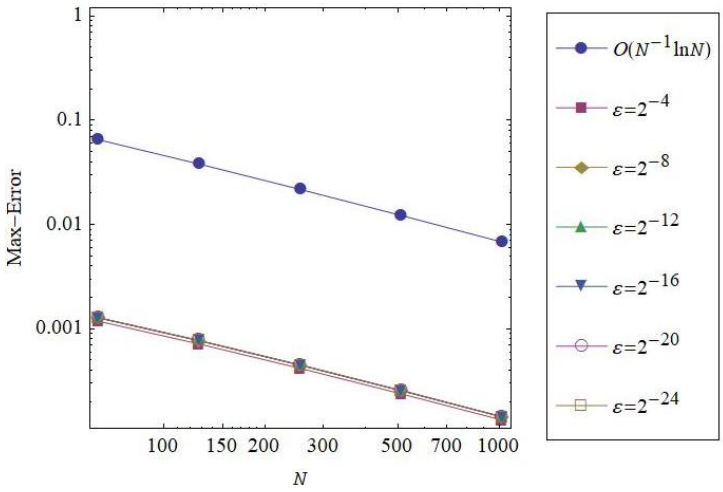


Figure 4. Maximum point-wise errors obtained using various ε and N for Example 4.2.

The data in Tables 1-2 contain the numerical results obtained from Example 4.1 and Example 4.2 for various values of N and the perturbation parameter. These results demonstrate that, for u, as the number of mesh points (N) increase, the maximum two mesh differences (e^N) monotonically decrease and the convergence rates (p^N) monotonically increase. computational results of the problem using S-Mesh are shown in Figures 1-4. These results indicate that the solution u at $t = 0$ is the boundary layer. All these results imply that these errors are bounded by $O(N^{-1}lnN)$, and are proven in Theorem 4.1. Moreover, all results were obtained in less than 20 seconds (for each ε and N) using an average modern computer, which indicates reasonable computational efficiency.

5 CONCLUSION AND SUGGESTIONS

In this work, the periodical problem for the first order linear SPDE is dealt with. For the approximate solution to this problem, a novel fitted difference scheme was constructed on the S-mesh by using the finite difference method. The convergence and stability of the proposed numerical method have been discussed. Furthermore, for the test problems, the numerical results as the computed two mesh differences and the rates of convergence were given in the tables and figures. According to these results, it was observed that the scheme is stable and has convergence almost first order when using S-Mesh, which supported the theoretical results. From all these results, it was concluded that the suggested approach is effective and accurate. The offered method in this study can be applied to the numerical solution of nonlinear type singularly perturbed periodical, and the delay differential equations. If a nonlinear problem is considered, then it will be more difficult to evaluate the convergence results since the difference scheme is nonlinear. This can be examined as another subject of study.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

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